

# Hypocycloidal throat for 2+1-dimensional thin-shell wormholes

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Recently we have shown that for 2 + 1-dimensional thin-shell wormholes a non-circular throat may lead to a physical wormhole in the sense that the energy conditions are satisfied. By the same token, herein we consider angular dependent throat geometry embedded in a 2 + 1-dimensional flat spacetime in polar coordinates. It is shown that a generic, natural example of throat geometry is provided remarkably by a hypocycloid. That is, two flat 2 + 1-dimensions are glued together along a hypocycloid. The energy required in each hypocycloid increases with the frequency of the roller circle inside the large one.

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## I. INTRODUCTION

Similar to the black holes the wormholes in 2 + 1-dimensions [1] also have certain degree of simplicity compared to their 3 + 1-dimensional counterparts [2]. The absence of gravitational degrees in 2 + 1-dimensions enforces us to introduce appropriate sources to keep the wormhole alive against collapse. Instead of general wormholes our concern will be confined herein to the subject of thin-shell wormholes (TSWs), whose throat is designed to host the entire source [3, 4]. From the outset our strategy will be to curve the geometry of the throat and find the corresponding energy-momentum through the Einstein's equations on the thin-shell [5, 6]. Clearly any distortion / warp at the throat gives rise to certain source, but as the subject is TSWs the nature of energy-density becomes of utmost important. Wormholes in general violates the null-energy condition (NEC) [7], which implies also the violation of the remaining energy conditions. The occurrence of negative pressure components in 3 + 1-dimensions provides alternatives in the sense that violation of NEC can be accounted by the pressure, leaving the possibility of an overall positive energy density.

In this paper we choose our throat geometry in 2 + 1-dimensional TSW such that the pressure vanishes, the energy density becomes positive and as a result all energy conditions are satisfied [5]. This is an advantageous situation in 2 + 1-dimensions not encountered in 3 + 1-dimensional TSWs. Our method is to consider a hypersurface induced in a 2 + 1-dimensional flat polar coordinates. Upon determining the energy density it is observed that a natural solution for the underlying geometry of the throat turns out to be a hypocycloid. Standard cycloid is known to be the minimum time curve of a falling particle under uniform gravitational field which is generated by a fixed point on a circle rolling on a straight

line. The hypocycloid on the other hand is generated by a fixed point on a small circle which rolls inside the circumference of a larger circle. The warped geometry of such a curve surprisingly generates energy density that turns out to be positive. This summarizes in brief, the main contribution of this paper.

In [5] we have constructed a 2 + 1-dimensional TSW by considering a flat bulk metric of the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \quad (1)$$

with a throat located at the hypersurface

$$F(r, \theta) = r - a_0(\theta) = 0. \quad (2)$$

Using the standard formalism of cut and paste technique (see the Appendix) it was shown that the line element of the throat is given by

$$ds_\Sigma^2 = -dt^2 + (a_0^2 + a_0'^2) d\theta^2 \quad (3)$$

with the energy momentum tensor on the shell

$$S_i^j = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & 0 \end{pmatrix} \quad (4)$$

in which

$$\sigma_0 = \frac{1}{4\pi} \frac{\left(a_0'' - a_0 - \frac{2a_0'^2}{a_0}\right)}{(a_0'^2 + a_0^2) \sqrt{1 + \left(\frac{a_0'}{a_0}\right)^2}}. \quad (5)$$

We note that a prime stands for the differentiation with respect to  $\theta$ . It was found that with  $\sigma_0 \geq 0$  all energy conditions are satisfied including that the matter which supports the wormhole was physical i.e. not exotic. Finally the total matter contained in the throat can be calculated as

$$U = \int_0^{2\pi} a_0 \sigma_0 d\theta. \quad (6)$$

In the sequel we shall give explicit examples for this integral.

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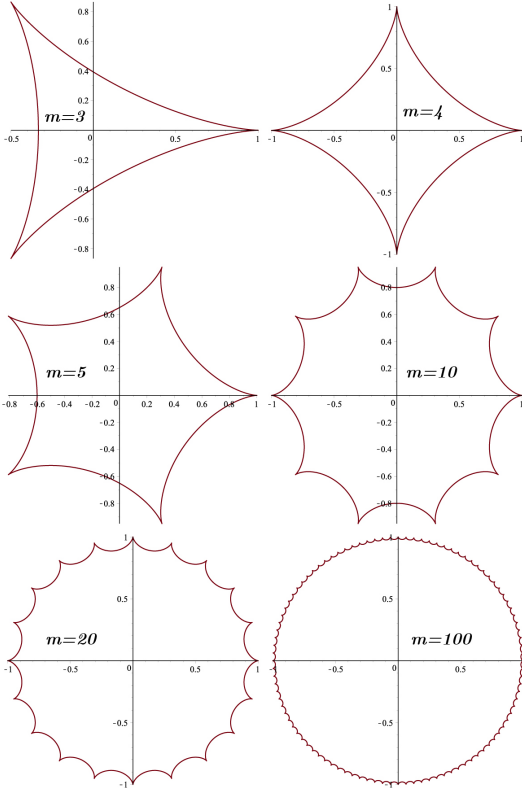


FIG. 1: Hypocycloid for different values of  $m = 3, 4, 5, 10, 20, 100$ .

## II. THE HYPOCYCLOID

Hypocycloid [8] is the curve generated by a rolling small circle inside a larger circle. This is a different version of the standard cycloid which is generated by a circle rolling on a straight line. The parametric equation of a hypocycloid is given by

$$\begin{aligned} x(\zeta) &= (B - b) \cos \zeta + b \cos \left( \frac{B - b}{b} \zeta \right) \\ y(\zeta) &= (B - b) \sin \zeta - b \sin \left( \frac{B - b}{b} \zeta \right) \end{aligned} \quad (7)$$

in which  $x$  and  $y$  are the Cartesian coordinates on the hypocycloid.  $B$  is the radius of the larger circle centered at the origin,  $b (< B)$  is the radius of the smaller circle and  $\zeta \in [0, 2\pi]$  is a real parameter. Here if one considers  $B = mb$ , where  $m \geq 3$  is a natural number, then the curve is closed and it possesses  $m$  singularities / spikes. In Fig. 1 we plot (7) for different values of  $m$  with  $B = 1$ . Let us add that for the particular choice of  $B = 1$  and  $b = \frac{1}{4}$  the hypocycloid takes a compact form  $x = \cos^3 \zeta$  and  $y = \sin^3 \zeta$  with  $x^{2/3} + y^{2/3} = 1$ . In what follows we proceed to determine the form of energy density  $\sigma$  and the resulting total energy for the individual cases plotted in Fig. 1.

To this end without loss of generality we set  $B = 1$  and  $b = \frac{1}{m}$  and express  $\sigma$  as a function of  $\zeta$ . For this we

parametrize the equation of the throat as

$$\begin{aligned} a &= a(\zeta) = \sqrt{x(\zeta)^2 + y(\zeta)^2} \\ \theta &= \theta(\zeta) = \tan^{-1} \left( \frac{y(\zeta)}{x(\zeta)} \right). \end{aligned} \quad (8)$$

Using the chain rule one finds

$$a' = \frac{da}{d\theta} = \frac{\dot{a}}{\dot{\theta}} \quad (9)$$

and

$$a'' = \frac{d^2 a}{d\theta^2} = \frac{\ddot{a}\dot{\theta} - \dot{a}\ddot{\theta}}{\dot{\theta}^3} \quad (10)$$

which implies

$$\sigma = \frac{1}{4\pi} \frac{a\ddot{a}\dot{\theta} - a\dot{a}\ddot{\theta} - a^2\dot{\theta}^3 - 2\dot{\theta}\dot{a}^2}{\left(\dot{a}^2 + a^2\dot{\theta}^2\right)^{\frac{3}{2}}} \quad (11)$$

where a dot stands for the derivative with respect to the parameter  $\zeta$ . Consequently the total matter is given by

$$U = \int_0^{2\pi} u d\zeta \quad (12)$$

where  $u = a\sigma\dot{\theta}$  is the energy density per unit parameter  $\zeta$ . Note that for the sake of simplicity we dropped the sub-index 0 from the quantities calculated at the throat. Particular examples of calculations for the energy  $U$  are given as follows.

### A. $m = 3$

The first case which we would like to study is the minimum index for  $m$  which is  $m = 3$ . We find that

$$\sigma = \frac{3\sqrt{2}}{32\pi\sqrt{(1 + 2\cos\zeta)^2(1 - \cos\zeta)}} \quad (13)$$

which is clearly positive everywhere. Knowing that the period of the curve (7) is  $2\pi$  we find that  $\sigma$  is singular at the possible roots of the denominator i.e.,  $\zeta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ . We note that although  $\sigma$  diverges at these points the function that must be finite everywhere is  $u$  which is given by

$$u = \frac{3\sqrt{2}\sqrt{(1 + 2\cos\zeta)^2(1 - \cos\zeta)}}{16\pi\sqrt{5 - 12\cos\zeta + 16\cos^3\zeta}}. \quad (14)$$

The situation is in analogy with the charge density of a charged conical conductor whose charge density at the vertex of the cone diverges while the total charge remains finite. In Fig. 2 we plot  $\sigma$  and  $u$  as a function of  $\zeta$  which clearly implies that  $u$  is finite everywhere leading to the total finite energy  $U_3 = 0.099189$ .

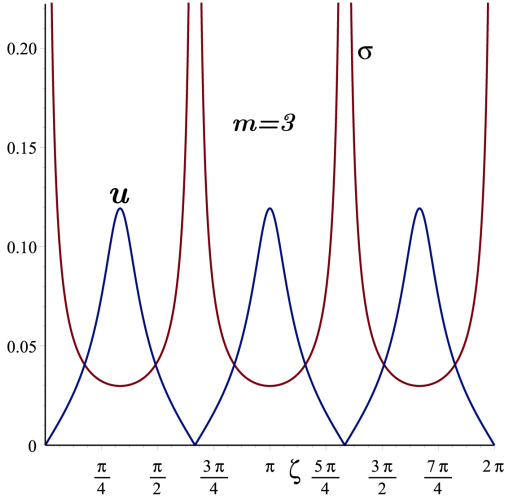


FIG. 2: Plot of  $\sigma$  and  $u$  in terms of  $\zeta$  for  $m=3$ . The singularities / cusps of  $\sigma$  are not physical. This can be seen easily when the total energy is finite. This is in analogy with a conical conductor of total charge finite but the charge density at the vertex diverges.

We would like to add that physically nothing extraordinary happens at the cusp points. These points are the specific points at which the manifold is not differentiable just with respect to  $r$  but also with respect to the angular variable  $\theta$ . The original thin-shell wormhole has been constructed based on discontinuity of the manifold with respect to  $r$  at the location of the throat which implied the presence of the matter source at the throat (We refer to the Fig. 1 of Ref. [9] where clearly such cuspy point in  $r$  direction is shown). Now, in the case under study we have one additional discontinuity of the Riemann tensor in  $\theta$  direction which implies a more complicated form of matter distribution at the throat.

### B. $m=4$

Next, we set  $m=4$  where one finds

$$\sigma = \frac{1}{6\pi\sqrt{\sin^2(2\zeta)}}, \quad (15)$$

and

$$u = \frac{\sqrt{\sin^2(2\zeta)}}{8\pi\sqrt{1-3\cos^2\zeta+3\cos^4\zeta}}. \quad (16)$$

Fig. 3 depicts  $\sigma$  and  $u$  in terms of  $\zeta$  and similar to  $m=3$ , we find  $\sigma > 0$  and  $u$  finite with the total energy given by  $U_4 = 0.24203$ . As one observes  $U_4 > U_3$  which implies that adding more cusps to the throat increases the energy needed. This is partly due to the fact that the total length of the hypocycloid is increasing as  $m$  increases

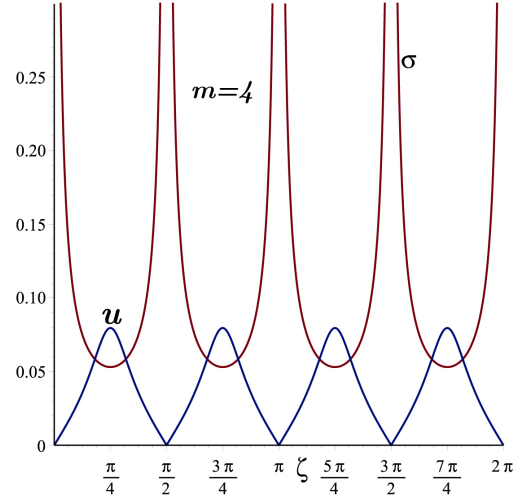


FIG. 3: Plot of  $\sigma$  and  $u$  in terms of  $\zeta$  for  $m=4$ . Similar to  $m=3$  the singularities of  $\sigma$  are not physical.

such that  $\ell_m = \frac{8(m-1)}{m}$  with  $B=1$ . This pattern goes on with  $m$  larger and in general

$$u = \frac{(m-2)^2 \sqrt{(\cos\zeta - \cos(m-1)\zeta)^2}}{8\pi\sqrt{2}\sqrt{\Psi}} \quad (17)$$

where

$$\begin{aligned} \Psi = & m^2 - 2(m-1)\cos^2(m-1)\zeta \\ & - (m-2)^2\cos\zeta\cos(m-1)\zeta - \\ & m^2\sin(m-1)\zeta\sin\zeta - 2(m-1)\cos^2\zeta. \end{aligned} \quad (18)$$

Table 1 shows the total energy  $U_m$  for various  $m$ . We observe that  $U_m$  is not bounded from above (with respect to  $m$ ) which means that for large  $m$  it diverges as  $U_m \approx \frac{m}{2\pi}$ . Therefore to stay in classically finite energy region one must consider  $m$  to be finite.

TABLE I: Total energy  $U_m$  in terms of  $m$ . We add that the total energy for large  $m$  is approximately given by  $U_{m \rightarrow \text{large}} \approx \frac{m}{2\pi}$ . This shows that increasing the number of cusps to infinity requires infinite energy.

$m$	3	4	5	10	50	100
$U_m$	0.099189	0.24203	0.39341	1.1767	7.5351	15.492

### III. CONCLUSION

The possibility of total positive energy has been scrutinized and verified with explicit examples in the 2+1-dimensional TSWs. Naturally the same subject arises with more stringent conditions in the more realistic dimensions of 3+1. By getting advantage of technical simplicity we have shown that the geometry of the throat can

remarkably be that of a hypocycloid. This is a rare curve compared with the more familiar minimum time cycloid. In effect, a fixed point on the circumference of a smaller circle rolling in a larger one makes the hypocycloid. The important point is that in the rolling process concavity of the resulting curve makes the extrinsic curvature negative, which in turn yields a positive energy density  $\sigma$ . Note that with convex curves this is not possible. The emerging cusps at the tips of the hypocycloid may yield singular points, however, these can be overcome by integrating around such cusps. The lightning rod analogy for diverging charge density in electromagnetism constitutes an example to understand the situation. In the present case our sharp points (edges) are reminiscent of cosmic strings and naturally deserves a separate investigation. The results for the total energy turns out to be perfectly positive, as our analytical calculation and numerical plots reveal. Increasing frequency of each roll by using smaller and smaller circles inside the large one is shown to increase the regular energy in finite amounts which is necessary to give life to a TSW. The fact that in a static frame the pressure vanishes simplifies our task. Here once  $\sigma > 0$  is chosen it implies automatically that the energy conditions are also satisfied. Finally, gluing together two curved spaces instead of flats will be our next project to address in the same line of thought.

## Appendix A: Extrinsic curvature tensor

The bulk metric is flat given by (1), therefore we cut out  $r < a(\theta)$  from the bulk and make two identical copies of the rest manifold. We paste them at the timelike hypersurface  $F(r, \theta) = r - a(\theta) = 0$  to construct a com-

plete manifold. The induced metric on the hyperplane  $\Sigma$  is given by (3). The extrinsic curvature tensor on the shell  $\Sigma$  is given by

$$K_{ij}^{(\pm)} = -n_\gamma^{(\pm)} \left( \frac{\partial^2 x^\gamma}{\partial y^i \partial y^j} + \Gamma_{\alpha\beta}^\gamma \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} \right) \quad (\text{A1})$$

in which  $x^\gamma = (t, r, \theta)$  is the coordinate of the bulk metric and  $y^i = (t, \theta)$  is the coordinate of the shell. Also

$$n_\gamma^{(\pm)} = \frac{\pm 1}{\sqrt{\Delta}} \frac{\partial F}{\partial x^\gamma} \quad (\text{A2})$$

where

$$\Delta = g^{\alpha\beta} \frac{\partial F}{\partial x^\alpha} \frac{\partial F}{\partial x^\beta} \quad (\text{A3})$$

refers to the normal 3-vector to the shell and  $\pm$  implies the different sides of the shell.

The Israel junction [10] conditions read

$$-8\pi S_i^j = k_i^j - \delta_i^j k \quad (\text{A4})$$

in which  $S_i^j = \text{diag}(-\sigma, p)$  is the energy-momentum tensor on the shell (we note that the off diagonal term is zero) and  $k_i^j = K_i^{j(+)} - K_i^{j(-)}$  with  $k = k_i^i$ . The explicit calculation reveals that

$$n_\gamma^{(\pm)} = \pm \frac{a}{\sqrt{a^2 + a'^2}} (0, 1, -a') \quad (\text{A5})$$

and

$$k_i^j = \begin{bmatrix} 0 & 0 \\ 0 & \frac{2(a^2 + 2a'^2 - aa'')}{(a^2 + a'^2)^{3/2}} \end{bmatrix}. \quad (\text{A6})$$

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